NAME: Solution

2.5 Author rigorous proofs of properties of graphs and their associated algorithms.

A graph $G = (V, E)$ is *bipartite* if its vertices can be divided into two disjoint sets V_1, V_2 (i.e., $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$) such that every edge of E connects a vertex in V_1 and a vertex in V_2.

A tree is an acyclic graph, i.e., a graph that contains no cycles. Recall that all trees can be rooted, where all edges represent a parent-child relationship.

Prove the following claim (justify in a few well argued sentences using these definitions).

Claim. *All trees are bipartite graphs.*

Consider a rooted tree, and let the 'level' of each vertex be the number of edges in the path between it and the root (a tree is acyclic, so there is only one such path).

Note that a vertex v at level i will have a parent at level $i - 1$ (since the parent is one edge closer to the root), and children at level $i + 1$ (since the children are one edge further than the root).

If i is even, then all vertices that are adjacent to v are on odd levels: $i - 1$ and $i + 1$. And if i is odd, then all vertices adjacent to v are on even levels.

So we can partition our graph into two sets: V_1 will be the set of all vertices at an even level in the tree, and V_2 will be the set of all vertices at an odd level in the tree.

All vertices in the tree are either on an even or an odd level, so $V_1 \cup V_2 = V$. Also note that no vertex is on both an odd and even level, so $V_1 \cap V_2 = \emptyset$. Moreover, we have already shown that every vertex in V_1 is only adjacent to vertices in V_2 and vice versa!

Thus the tree must be a bipartite graph.

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